PHYS5150 — PLASMA PHYSICS

LECTURE 16 - RESISTIVITY

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1 RESISTIVITY

RESISTIVITY E field accelerate electrons until collisional friction with ions is large enough to counteract it – electrons reach terminal velocity similar to parachute.

DIFFUSION Collisions with electrons moves the guiding center, so that particle can diffuse from one field line to another.

2 HARD SPHERE COLLISIONS

$$n = \frac{1}{V} = \frac{1}{\sigma \cdot s} = \frac{1}{\sigma v_{th} \Delta t}$$

and thus,

$$\frac{1}{\Delta t} = \gamma = n \cdot \sigma \cdot v_{th}.$$

In the general case

$$\boxed{\gamma_{coll} = \frac{1}{\tau_{coll}} = n \langle \sigma \cdot v \rangle}$$

3 ELECTRON-ION-“COLLSIONS”

We consider the effect of an motionless ion onto the trajectory of an approaching electron:

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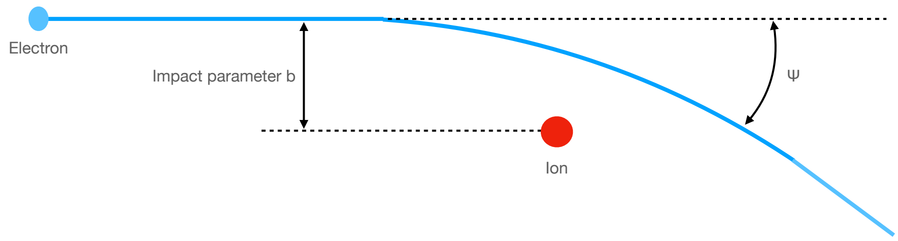
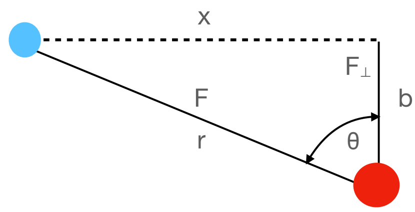


We can derive the dependence of the deflection angle Ψ on the impact parameter b for general potentials, but here it is sufficient to solve it for ions and electrons:



Using the geometry shown above the force acting on the electron is

$$F = \frac{e^2}{4\pi\epsilon_0 r^2} = \frac{k}{r^2},$$

and its perpendicular component is

$$F_\perp = F\cos\theta = \frac{k}{r^2}\cos\theta.$$

Using $r = \frac{b}{\cos\theta}$ we get

$$F_\perp = \frac{k}{b^2}\cos^3\theta$$

and the corresponding perpendicular acceleration and velocity are

$$a_\perp = \frac{F_\perp}{m} = \frac{k}{mb^2}\cos^3\theta = \frac{\mathrm{d}v_\perp}{\mathrm{d}t},$$

and

$$v_\perp = \int \frac{\mathrm{d}v_\perp}{\mathrm{d}t}dt = \frac{k}{mb^2}\int \cos^3\theta dt.$$

Now, how does θ vary with t? With

$$\tan\theta = \frac{x}{b} \to x = b\tan\theta,$$

we find that

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{b}{\cos^2\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

or

$$dt = \frac{b}{v\cos^2\theta}d\theta.$$

The perpendicular velocity after collision computes as

$$v_\perp = \frac{k}{mb^2}\int_{-\pi/2}^{\pi/2}\cos^3\theta dt = \frac{k}{mb^2}\frac{b}{v}\underbrace{\int_{-\pi/2}^{\pi/2}\cos\theta d\theta}_{2} = \frac{2k}{mbv},$$

and after replacing k

$$v_\perp = \frac{e^2}{2\pi\epsilon_0 mbv}.$$

We can now determine the scatter angle

$$\sin\Psi = \frac{v_\perp}{v},$$

and for small scatter angles, i.e. $\sin\Psi \approx \Psi$,

$$\boxed{\Psi = \frac{e^2}{2\pi\epsilon_0 mbv^2}.}$$

Large angle collision occurs if $v_\perp/v \approx 1$:

$$\boxed{b_0 = \frac{e^2}{2\pi\epsilon_0 mv^2}.}$$

Furthermore, $\Psi = \frac{b}{b_0}$ for $\Psi \ll 1$.

Many small angle angular deflections accumulate to make a large angle collision. Because Ψ changes randomly, the electron performs a *random walk*:

$$\begin{aligned}\langle\, \Psi^2 \,\rangle &= \langle\, \Psi_1^2 \,\rangle + \langle\, \Psi_2^2 \,\rangle + \langle\, \Psi_3^2 \,\rangle + \dots \\ &= \underbrace{\tfrac{1}{2}}_{\text{2D}}\ \underbrace{N}_{\text{collision \#}}\ \underbrace{\langle\, d\Psi^2 \,\rangle}_{\text{mean deflection angle}} \quad .\end{aligned}$$

Now, to figure out $\langle\, d\Psi^2 \,\rangle$, we need to know number of collisions per time for fixed b

$$\frac{\langle\, \Psi^2 \,\rangle \text{ increase}}{\text{unit length}} = \sum (d\Psi^2 \text{ at each } b) \times \frac{N \text{ at each b}}{\text{unit length}} \times \tfrac{1}{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}L} \langle\, \Psi^2 \,\rangle = \tfrac{1}{2} \int \left(\frac{b_0}{b}\right)^2 n 2\pi b\, db$$

$$= n\pi b_0^2 \int_{b_{min}}^{b_{max}} \frac{db}{b} = n\pi b_0^2 \ln\left(\frac{b_{max}}{b_{min}}\right)$$

$\mathbf{b_{min}}$: for summing over small angles $\Psi \leq 1$, i.e. $b_{min} = b_0$

$\mathbf{b_{max}}$: Debye shielding cuts off Coulomb force at $b_{max} \sim \lambda_D = \left(\frac{\epsilon_0 \mathrm{k_B} T}{ne^2}\right)^{1/2}$

$$\frac{b_{max}}{b_{min}} = \frac{\lambda_D}{b_0} = \left[\frac{\epsilon_0 \mathrm{k_B} T}{ne^2}\right]^{1/2} \left[\frac{2\pi\epsilon_0 m v^2}{e^2}\right],$$

using $\mathrm{k_B} T = \frac{1}{2} m v^2$

$$\frac{b_{max}}{b_{min}} = \left[\frac{\epsilon_0 \mathrm{k_B} T}{ne^2}\right]^{1/2} \left[\frac{\epsilon_0 \mathrm{k_B} T}{e^2}\right] 4\pi \left(\frac{n}{n}\right)$$

$$\frac{b_{max}}{b_{min}} = 4\pi n \lambda_D^3,$$

and with

$$N_D = \frac{4}{3}\pi n \lambda_D^3$$

we get

$$\frac{b_{max}}{b_{min}} = 3N_D = \Lambda.$$

Introduce the *Coulomb logarithm*

$$\ln \frac{b_{max}}{b_{min}} = \ln \Lambda.$$

Now,

$$\frac{\langle\, \Psi^2 \,\rangle \text{ increase}}{\text{unit length}} = \pi n b_0^2 \ln \Lambda.$$

Define λ_{mfp} to be the typical length for an electron to acquire large angle collision $\langle\, \Psi^2 \,\rangle = 1$ from small angle collisions

$$\frac{\langle\, \Psi^2 \,\rangle \text{ increase}}{\text{unit length}} = \frac{1}{\lambda_{mfp}} = \pi n b_0^2 \ln \Lambda.$$

Obvioulsy

$$\lambda_{mfp} = \frac{1}{\pi n b_0^2 \ln \Lambda}$$

has the meaning of a *mean free path length* for large angle collisions. Now recall that we found for hard sphere collisions that the mean free path is $(n\sigma)^{-1}$, and thus

$$n\sigma = n\pi b_0^2 \ln \Lambda.$$

From this follows that the cross section for *Coulomb collisions* is

$$\sigma = \pi b_0^2 \ln \Lambda.$$

After substituting back $b_0 = e^2/2\pi\epsilon_0 \mathrm{k_B} T$ we get the final expression for the cross section

$$\boxed{\sigma = \pi \left(\frac{e^2}{2\pi\epsilon_0 \mathrm{k_B} T} \right)^2 \ln \Lambda.}$$

Note that σ is independent of the plasma density and scales with temperature as T^{-2}.